

Wang tilings problem of $m \times n$ rectangles with monochromatic edges and a specific tile set $\mathcal{T}$

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Abstract

In this paper, we study the tiling of $m \times n$ rectangles with monochromatic edges using a particular set $\mathcal{T}$ of Wang tiles of four colors. We distinguish three types of rectangle according to the colors assigned to their opposite edges. For each type, we give necessary and sufficient conditions on m and n for the existence of a $\mathcal{T}$ -tiling. The sufficient conditions are established through explicit constructions and an induction argument, whereas the necessary conditions follow from a parity invariant. We then use this classification to determine the minimum number of colors required to tile $m \times n$ rectangles of every type and all dimensions $m, n \geq 2$, and prove that this minimum is five colors. Finally, we propose several friendly problems for further investigation, ranging from determining the minimum number of tiles needed for tiling to studying analogous problems in dimensions $d \geq 3$.

Keywords: Domino Problem, Wang tiles, Rectangles with monochromatic edges, Discrete mathematics

1 A short story about the Domino Problem

The Domino Problem was introduced in the 1960s by Wang [1] in connection with problems in first-order logic with the aim of engaging a larger community of mathematicians [2]. It asks whether, for a given finite tile set $\mathcal{T}$, the discrete plane $\mathbb{Z}^2$ can be tiled using tiles from $\mathcal{T}$, subject to the prescribed edge-matching rules. A Wang tile is a unit square with a color assigned to each of its four edges. It can be depicted as four colored triangles meeting at the center of the square, each triangle

indicating the color of one edge. Tiles may not be rotated or reflected. Two tiles can be matched if and only if they share an edge of the same color (Fig. 1).

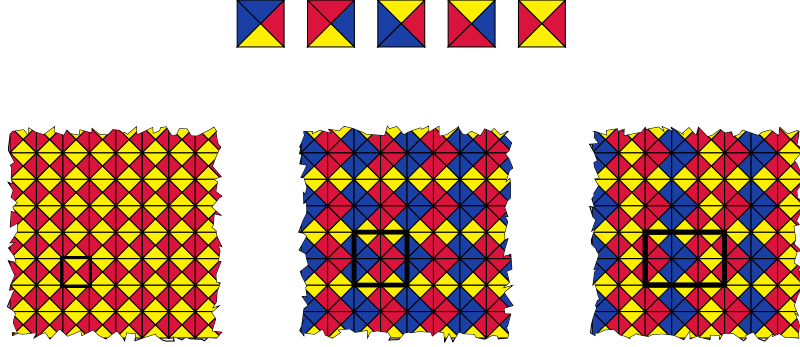


Fig. 1: Example of a set $\mathcal{T}$ of five Wang tiles and three periodic tilings of $\mathbb{Z}^2$ using tiles from $\mathcal{T}$.

Wang initially conjectured that every Wang tile set that tiles the plane $\mathbb{Z}^2$ admits a periodic tiling. Berger, Wang's student, refuted this conjecture in 1966 and showed that there existed a Wang tile set which tiles the plane but admits no periodic tiling (aperiodic tiling) and also proved the undecidability of the problem. This tile set contained 20 426 tiles. He then reduced this number to 104 tiles [3, 4]. Several mathematicians have since reduced this number using different approaches such as discrete geometry, plane geometry, arithmetic and algebraic methods and exhaustive computer searches, for instance. At the same time, Laüchli constructed an aperiodic tile set of 40 tiles. This tile set was not published until 1975 [5]. In the 1970s, Robinson discovered several aperiodic collections of 56 and 52 tiles [6, 7]. A complete description of the latter tile set was given in [8]. Robinson also mentioned another aperiodic set containing 35 tiles without further details [7]. In 1968, Knuth also reduced the number of tiles to 92 [9]. In the 1970s, this number was reduced to 32 and then 24 by Robinson and Penrose [10]. Ammann proposed another reduction to 16 tiles which was published in [10], [11]. Kari and Culik reduced this number to 14 and 13, respectively, in 1996 [12, 13]. Finally, Jeandel and Rao announced in 2015 [14] that the smallest aperiodic Wang tile set consists of 11 tiles using only four colors (Fig. 2).

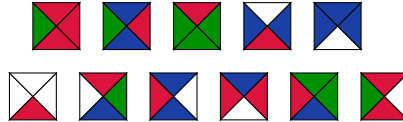


Fig. 2: Aperiodic tile set with 11 Wang tiles and 4 colors [14].

The Domino Problem has also been studied on finite regions of $\mathbb{Z}^2$ such as rectangles [15–18] or (bi)-infinite strips with the height either fixed or given as part of the input [19–21]. In particular, we have known since the 1970s that this problem is $\mathcal{NP}$ -complete on rectangles [15, 17]. Furthermore, when one or more edges of the

rectangle are required to be monochromatic, this problem remains $\mathcal{NP}$ -complete [20].

In this paper, we propose to explore the Wang tilings problem on $m \times n$ rectangles with monochromatic edges but using a funny choice of Wang tiles. This problem was studied in the author's doctoral thesis in mathematics education [20] to design research situations aimed at developing reasoning and proving processes in mathematics. Here, we focus on its mathematical aspects and show that the local color-matching constraints give rise to global parity conditions. With four colors, these conditions are both necessary and sufficient for a tiling to exist. Adding a fifth color makes it possible to tile rectangles that do not satisfy these conditions.

2 Wang tiling problem on $m \times n$ rectangles with monochromatic edges

Let $\mathcal{T}$ be a tile set and let $\mathcal{X}_{\mathcal{T}}$ be the color set of $\mathcal{T}$ such that $|\mathcal{X}_{\mathcal{T}}| = 4$. Each tile in $\mathcal{T}$ has each of these four colors exactly once (e.g., blue, yellow, green, red). Moreover, all permutations are allowed. So, we have $|\mathcal{T}| = 24$ tiles. The question is whether, for any integers $m, n \geq 1$, there exists a tiling of an $m \times n$ rectangle with tiles of $\mathcal{T}$ whose sides are monochromatic.

As each tile always has four different colors, adjacent sides of the rectangle must have two different colors. Therefore, there are three different types of rectangles to consider, subject to some initial conditions on the integers $m \geq 1$ and $n \geq 1$ (Fig. 3). We use the following notation:

- $\mathcal{R}_{m \times n}^{\mathfrak{A}}$ a rectangle of $\mathfrak{A}$ -type with two opposite edges two by two of the same color;
- $\mathcal{R}_{m \times n}^{\mathfrak{B}}$ a rectangle of $\mathfrak{B}$ -type with the east-west edges of the same color;
- $\mathcal{R}_{m \times n}^{\mathfrak{C}}$ a rectangle of $\mathfrak{C}$ -type with the north-south of the same color;
- $\mathcal{R}_{m \times n}^{\mathfrak{D}}$ a rectangle of $\mathfrak{D}$ -type with four monochromatic sides of distinct colors.

Moreover, for every type of rectangles, $\mathcal{R}_{m \times n}$ is called $\mathcal{T}$ -tileable if it admits a tiling using tiles from $\mathcal{T}$.

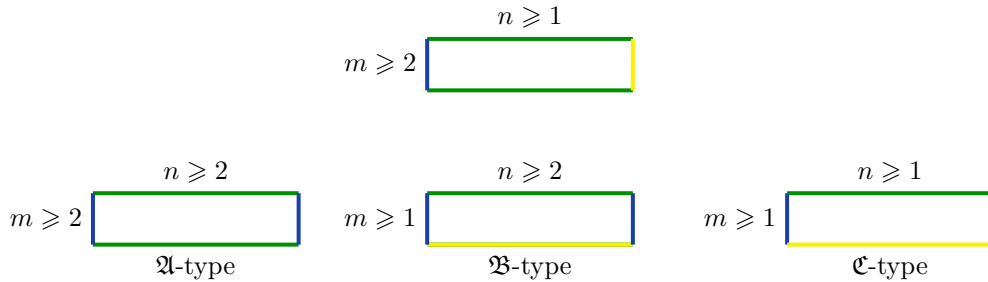


Fig. 3: Different types of rectangle.

Remark 1. Since $\mathcal{T}$ contains every tile with four pairwise distinct edge colors, it is invariant under permutations of $\mathcal{X}_{\mathcal{T}}$ and symmetries of the square. Thus, a tiling constructed for one boundary coloring also gives a tiling for every coloring of the same type, up to these symmetries.

We will see that the existence of a tiling depends, in fact, only on the parities of m and n and on the rectangle type. The following theorem gives a complete characterization of the tileable cases. We will prove it in the following sections.

Theorem 1. *Let $\mathcal{T}$ be the tile set under consideration.*

- (i) *Let $m, n \geq 2$. $\mathcal{R}_{m \times n}^{\mathfrak{A}}$ is $\mathcal{T}$ -tileable $\Leftrightarrow mn \equiv 0 \pmod{2}$.*
- (ii) *Let $m \geq 2$ and $n \geq 1$. $\mathcal{R}_{m \times n}^{\overline{\mathfrak{B}}}$ is $\mathcal{T}$ -tileable $\Leftrightarrow m \in 2\mathbb{N}$.*
- (iii) *Let $m \geq 1$ and $n \geq 2$. $\mathcal{R}_{m \times n}^{|\mathfrak{B}|}$ is $\mathcal{T}$ -tileable $\Leftrightarrow n \in 2\mathbb{N}$.*
- (iv) *Let $m, n \geq 1$. $\mathcal{R}_{m \times n}^{\mathfrak{C}}$ is $\mathcal{T}$ -tileable $\Leftrightarrow m \equiv n \pmod{2}$.*

Note that $\mathcal{T} = \emptyset$ when $|\mathcal{X}_{\mathcal{T}}| < 4$. With four colors, some rectangles cannot be tiled. We will then show that adding a fifth color ($|\mathcal{X}_{\mathcal{T}}| = 5$ gives $|\mathcal{T}| = 120$ tiles) suffices to tile every rectangle of any type with $m, n \geq 2$, establishing the following optimal bound.

Theorem 2. *Let $\mathcal{X}_{\mathcal{T}}$ be a finite set of colors and let $\mathcal{T}$ be the set of all Wang tiles whose four edges carry pairwise distinct colors from $\mathcal{X}_{\mathcal{T}}$. The minimum number of colors of $\mathcal{X}_{\mathcal{T}}$ required to tile every $m \times n$ rectangle of any type, with $m, n \geq 2$ is:*

$$\min |\mathcal{X}_{\mathcal{T}}| = 5.$$

3 From parity conditions on m and n to tilings

The study of particular cases with a small height $m \in \{1, 2, 3\}$ allows us to obtain initial results on the existence or non-existence of tilings.

Example 1. When $m = 1$, rectangles of type $\mathfrak{A}$ or $\overline{\mathfrak{B}}$ are clearly not tileable for any $n \geq 1$, since no tile has two edges of the same color. For rectangles of type $|\mathfrak{B}|$ or $\mathfrak{C}$, repeating a 1×2 pattern of type $|\mathfrak{B}|$ yields a $\mathcal{T}$ -tiling of type $|\mathfrak{B}|$ when n is even. Removing the last tile yields a $\mathcal{T}$ -tiling of type $\mathfrak{C}$ when n is odd. These conditions are also necessary, since the prescribed edge colors of the rectangle force the tiles to alternate according to this 1×2 pattern.

Example 2. For a $2 \times n$ rectangle of type $\mathfrak{C}$ with $n \geq 2$ even, we tile the first $n - 2$ columns by repeating a 2×2 pattern of $|\mathfrak{B}|$ -type with matching east and west edge colors. We then add a 2×2 pattern of type $\mathfrak{C}$ with compatible boundary colors to complete the tiling. Moreover, when $m = 2$ and $n \geq 1$ is odd, we prove by contradiction that no $\mathcal{T}$ -tiling exists. Assume that such a tiling exists. The boundary conditions impose yellow along the south edge and green along the north edge. Consequently, each tile in the second row (top row) must have exactly one yellow edge: either east or west. Since the west edge of the first tile is blue, its east edge must be yellow. The matching conditions then force the yellow edges to alternate

along the row. Since n is odd, the last tile must have a yellow east edge. This contradicts the boundary condition for a type- $\mathfrak{C}$ rectangle. Hence, no valid $\mathcal{T}$ -tiling of $2 \times n$ rectangle of $\mathfrak{C}$ -type exists when n is odd.

Example 3. For a $3 \times n$ rectangle of type $\mathfrak{A}$ with $n \geq 2$ even, repeating a 3×2 pattern of the same type horizontally yields a tiling. Furthermore, when $m = 3$ and $n \geq 1$ is odd, it is easy to show that no valid tiling of the rectangle of type $\mathfrak{A}$ exists (Fig. 4). Indeed, suppose, for contradiction, that a valid tiling of the $3 \times n$ rectangle of type- $\mathfrak{A}$ exists with n odd. Because of the color green already imposed by the tiles in the first and third rows, each tile in the middle row must have a green east or west edge. Moreover, the first tile of the middle row must have its blue west edge and therefore its green east edge. Hence, the green edges alternate along the row: the second tile must have a green west edge, the third a green east edge, and so on. Since n is odd, the last tile in the middle row is forced to have a green east edge rather than a blue one. This contradicts the boundary condition imposed on a rectangle of type $\mathfrak{A}$. Therefore, no valid $\mathcal{T}$ -tiling of a $3 \times n$ rectangle of type $\mathfrak{A}$ exists when n is odd.

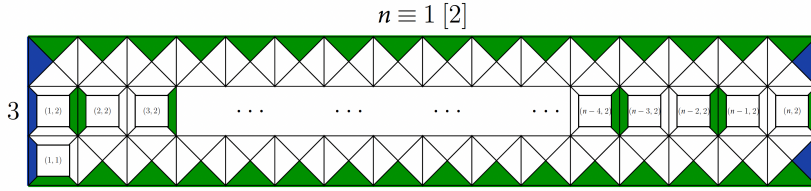


Fig. 4: No $\mathcal{T}$ -tiling of $\mathcal{R}_{3 \times n}^{\mathfrak{A}}$ exists when n is odd.

We have seen that a direct argument by contradiction can provide necessary conditions in some special cases, but it remains very limited. Indeed, when $m > 3$, this argument becomes increasingly difficult, as the number of subcases increases rapidly. The same difficulty arises for the other rectangle types, even when their dimensions are relatively small (for instance the case of 3×4 rectangle of $\mathfrak{C}$ -type). We will therefore adopt a different approach, counting the triangles of each color in two different ways to derive a parity invariant and establish the necessary conditions in Theorem 1.

Before developing this counting argument, we establish the sufficient conditions, described by the following proposition, through explicit constructions. We first justify the removal of outermost rows and columns used in these constructions.

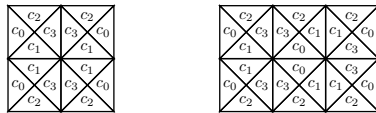


Fig. 5: Two basic patterns of $\mathfrak{A}$ -type with monochromatic vertical cuts.

Lemma 1. *By stacking aligned copies of either the 2×2 or the 2×3 pattern of $\mathfrak{A}$ -type as shown in Fig. 5, we obtain tilings in which every cut between columns is monochromatic. The same applies to cuts between rows, up to rotation.*

Proof. To construct these tilings, we start with two basic $\mathfrak{A}$ -type patterns of sizes 2×2 and 2×3 (Fig. 5). In both patterns, the east and west edges have color c_0 , and the north and south edges have color c_2 . Copies can therefore be repeated periodically in either direction. The vertical cut in the 2×2 pattern has color c_3 , while the two vertical cuts in the 2×3 pattern have colors c_3 and c_1 . When copies of either pattern are stacked, corresponding cuts align to form monochromatic vertical lines. Removing an outermost column therefore exposes a monochromatic edge. The analogous construction for rows follows by rotation. $\square$

Proposition 1. *Let $\mathcal{T}$ be the tile set and let m, n be integers.*

- (i) *Let $m, n \geq 2$. If mn is even, then $\mathcal{R}_{m \times n}^{\mathfrak{A}}$ is $\mathcal{T}$ -tileable.*
- (ii) *Let $m \geq 2$ and $n \geq 1$. If $m \in 2\mathbb{N}$, then $\mathcal{R}_{m \times n}^{\overline{\mathfrak{B}}}$ is $\mathcal{T}$ -tileable.*
- (iii) *Let $m \geq 1$ and $n \geq 2$. If $n \in 2\mathbb{N}$, then $\mathcal{R}_{m \times n}^{|\mathfrak{B}|}$ is $\mathcal{T}$ -tileable.*
- (iv) *Let $m, n \geq 1$. If $m \equiv n \pmod{2}$, then $\mathcal{R}_{m \times n}^{\mathfrak{C}}$ is $\mathcal{T}$ -tileable.*

Proof. We consider each type of rectangle separately.

Case 1: $\mathfrak{A}$ -type. We can easily construct a tiling $\mathcal{R}_{2 \times 2}^{\mathfrak{A}}$. From $\mathcal{R}_{2 \times 2}^{\mathfrak{A}}$, which has the same color on the east-west edges and the same color on the south-north edges, it suffices to repeat it horizontally $n/2$ times. The new tiling of size $2 \times n$ can then be repeated vertically $m/2$ times. Thus, we can tile the rectangle $\mathcal{R}_{m \times n}^{\mathfrak{A}}$ for all even integers $m, n \geq 2$ (Fig. 6).

Starting from a tiling of $\mathcal{R}_{m \times n}^{\mathfrak{A}}$, it suffices to delete the last two columns, we then get a tiling $\mathcal{R}_{m \times (n-2)}^{\mathfrak{A}}$ by Lemma 1. Then, we just need to construct $\mathcal{R}_{2 \times 3}^{\mathfrak{A}}$, stack $m/2$ copies vertically, and join the two parts together to tile the rectangle $\mathcal{R}_{m \times n'}^{\mathfrak{A}}$ with $n' = n + 1$ odd and m even. Moreover, as $\mathcal{T}$ contains all tiles involving the four colors, a quarter-turn rotation of the tiling of $\mathcal{R}_{m \times n'}^{\mathfrak{A}}$ yields a tiling of $\mathcal{R}_{n' \times m}^{\mathfrak{A}}$, whose height n' is odd and whose length m is even (Fig. 6). Therefore, we have proved that if mn is even with $m, n \geq 2$, then $\mathcal{R}_{m \times n}^{\mathfrak{A}}$ is $\mathcal{T}$ -tileable. The assertion (i) of Proposition 1 has now been proved.

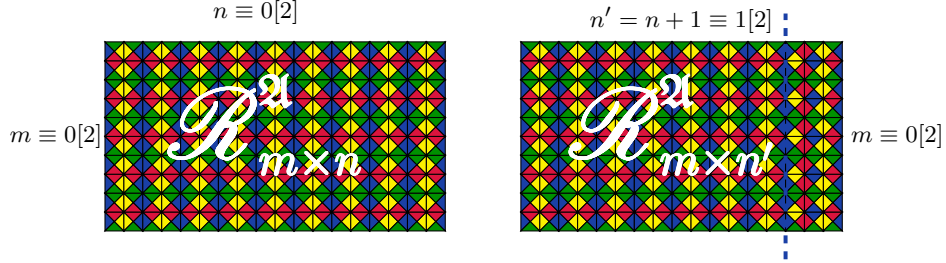


Fig. 6: Examples of $\mathcal{T}$ -tilings for $\mathfrak{A}$ -type rectangle.

Case 2: $\overline{\mathfrak{B}}$ -type and $|\mathfrak{B}|$ -type.

- (ii) From a $\mathcal{T}$ -tiling of $\mathcal{R}_{m \times n}^{\mathfrak{A}}$ with $m, n \geq 2$ both even, as established in (i), we can again, by Lemma 1, delete the last column n and therefore get a $\mathcal{T}$ -tiling of $\mathcal{R}_{m \times n'}^{\overline{\mathfrak{B}}}$ where $n' = n - 1 \geq 1$ is odd and $m \geq 2$ is even (Fig. 7).

We use the same argument with a $\mathcal{T}$ -tiling of $\mathcal{R}_{m \times n}^{\mathfrak{A}}$ when $m \geq 2$ is even and $n \geq 2$ is odd. By Lemma 1, we delete the last column, we have then a $\mathcal{T}$ -tiling of $\mathcal{R}_{m \times n''}^{\overline{\mathfrak{B}}}$ where $n'' = n - 1 \geq 1$ and $m \geq 2$ are even integers (Fig. 7). Therefore, if $m \in 2\mathbb{N}^*$ and $n \geq 1$, then there exists a $\mathcal{T}$ -tiling of $\mathcal{R}_{m \times n}^{\overline{\mathfrak{B}}}$.

- (iii) For the case of $|\mathfrak{B}|$ -type rectangle, we use a $\mathcal{T}$ -tiling of $\mathcal{R}_{m \times n}^{\mathfrak{A}}$ again, in particular when $m, n \geq 2$ are even but this time, we do not delete the last column n but the last row. By Lemma 1, we thus have a $\mathcal{T}$ -tiling of $\mathcal{R}_{m' \times n}^{|\mathfrak{B}|}$ with $m' = m - 1 \geq 1$ is odd and $n \geq 2$ is even. For the last case when m and n are both even, we can use assertion (ii), and in particular the $\mathcal{T}$ -tiling of $\mathcal{R}_{m \times n}^{\overline{\mathfrak{B}}}$ with m and n both even (Fig. 7). A quarter-turn rotation yields a $\mathcal{T}$ -tiling of $\mathcal{R}_{m \times n}^{|\mathfrak{B}|}$ with $m \in 2\mathbb{N}^*$ and $n \in 2\mathbb{N}^*$. Hence, if $m \geq 1$ and $n \in 2\mathbb{N}^*$, then $\mathcal{R}_{m \times n}^{|\mathfrak{B}|}$ admits a $\mathcal{T}$ -tiling.

This proves the assertions (ii) and (iii) of Proposition 1.

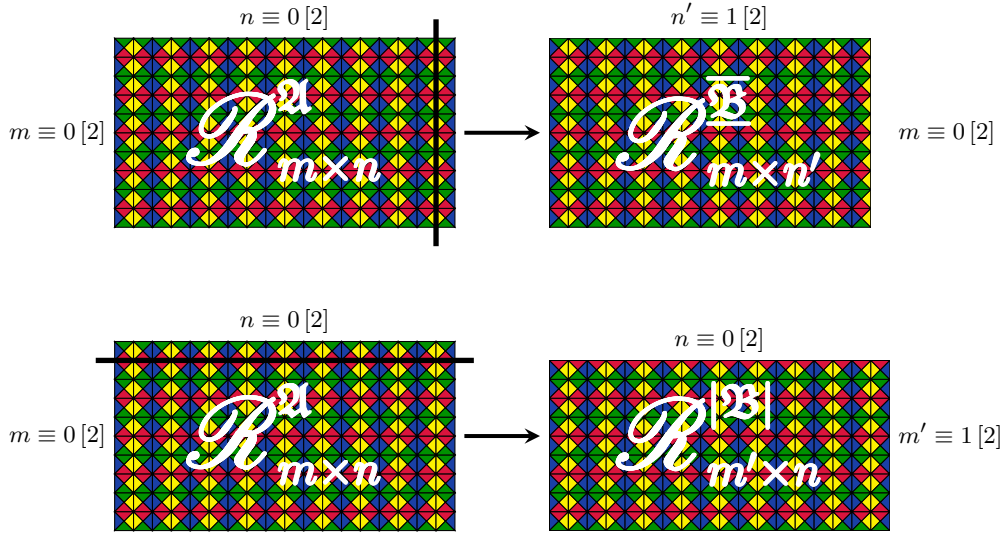


Fig. 7: Examples of $\mathcal{T}$ -tiling of the $|\mathfrak{B}|$ -type rectangle based on a tiling of $\mathfrak{A}$ -type.

Case 3: $\mathfrak{C}$ -type. Let us start with a $\mathcal{T}$ -tiling of $\mathcal{R}_{m \times n}^{\mathfrak{A}}$ given by (i), where $m, n \geq 2$ are both even. By deleting the last column and the last row, we get, thanks to Lemma 1, a $\mathcal{T}$ -tiling of $\mathcal{R}_{m' \times n'}^{\mathfrak{C}}$ where $m' = m - 1 \geq 1$ and $n' = n - 1 \geq 1$ are both odd (Fig. 8).

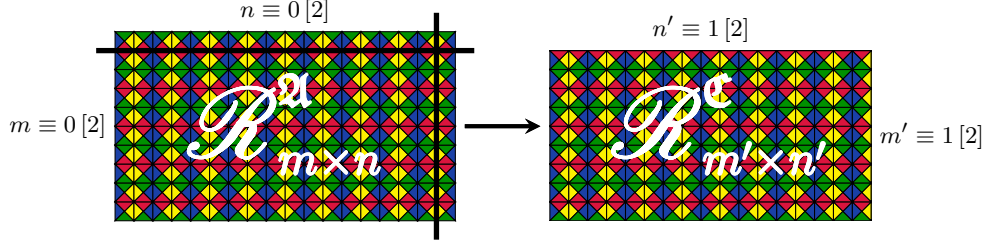


Fig. 8: Examples of $\mathcal{T}$ -tilings for the $\mathfrak{C}$ -type rectangle based on $\mathfrak{A}$ -type.

The case in which $m \geq 1$ and $n \geq 1$ are both even can also be established by decomposing the rectangle. Indeed, we can partition the $m \times n$ rectangle of type $\mathfrak{C}$ into four smaller rectangles of different sizes (Fig. 9). Furthermore, we know by (i) that $\mathcal{R}^{\mathfrak{A}}_{(m-2) \times (n-2)}$ is $\mathcal{T}$ -tileable when $m \geq 4$ and $n \geq 4$ are both even. Moreover, $\mathcal{R}^{|\mathfrak{B}|}_{2 \times (n-2)}$ and $\mathcal{R}^{\mathfrak{B}}_{(m-2) \times 2}$ are also both $\mathcal{T}$ -tileable when $m \geq 4$ and $n \geq 4$ thanks to (ii) and (iii) (see also the start of Example 2, up to rotation). Moreover, $\mathcal{R}^{\mathfrak{C}}_{2 \times 2}$ is clearly tileable with the tiles of $\mathcal{T}$ (see Fig. 9). The remaining cases have $m = 2$ or $n = 2$. They are tileable by Example 2, up to rotation, including the 2×2 case established above. This proves assertion (iv) of Proposition 1 for all even $m, n \geq 2$.

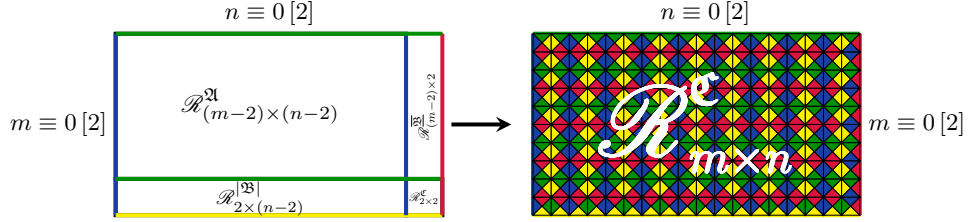


Fig. 9: An example of a partition of a $\mathfrak{C}$ -type rectangle.

□

Therefore, we have proved all the assertions of Proposition 1 and thus all the sufficient conditions of Theorem 1.

Remark 2. The proof of Proposition 1 uses two main periodic patterns of $\mathfrak{A}$ -type and of sizes 2×2 and 2×3 which enable the generation of most $\mathcal{T}$ -tilings by adding and removing rows and columns. An inductive approach also allows us to establish a general proof of all these sufficient conditions. We give the main idea of the proof below.

Sketch of proof by induction. Let $\mathcal{T}$ be the tile set under consideration and $\mathcal{X}_{\mathcal{T}} = \{c_0, c_1, \dots, c_3\}$ the set of four colors. We fix the colors c_0 and c_1 of $\mathcal{X}_{\mathcal{T}}$ on the west and south edges, respectively of the $m \times n$ rectangle (Fig. 10). Furthermore, all types of $m \times n$ rectangle can be partitioned into two smaller rectangles when m and n are sufficiently large. The first rectangle, of size $m \times (n - 2)$, is of the same type as the $m \times n$ rectangle and preserves the parities of both dimensions. The second rectangle has size $m \times 2$ and is of type $|\mathfrak{B}|$ or $\mathfrak{A}$. As shown in Figure 10, depending on the appropriate choice of colors X, Y and $Z \in \mathcal{X}_{\mathcal{T}}$, there are three distinct cases up to rotation.

- If $X = c_0, Y = c_1$ and $Z = c_0$, then we get an $\mathfrak{A}$ -type rectangle $\mathcal{R}_{m \times n}^{\mathfrak{A}}$ made up of two rectangles $\mathcal{R}_{m \times (n-2)}^{\mathfrak{A}}$ and $\mathcal{R}_{m \times 2}^{\mathfrak{A}}$
- If $X = c_0, Y = c_2$ and $Z = c_0$, then we get a $|\mathfrak{B}|$ -type rectangle $\mathcal{R}_{m \times n}^{|\mathfrak{B}|}$ made up of two rectangles $\mathcal{R}_{m \times (n-2)}^{|\mathfrak{B}|}$ and $\mathcal{R}_{m \times 2}^{|\mathfrak{B}|}$
- If $X = c_2, Y = c_3$ and $Z = c_2$, then we get a $\mathfrak{C}$ -type rectangle $\mathcal{R}_{m \times n}^{\mathfrak{C}}$ made up of two rectangles $\mathcal{R}_{m \times (n-2)}^{\mathfrak{C}}$ and $\mathcal{R}_{m \times 2}^{|\mathfrak{B}|}$

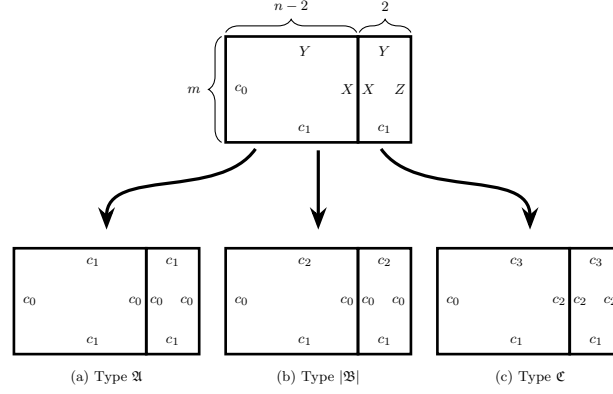


Fig. 10: Decomposition of an $m \times n$ rectangle into two smaller rectangles based on the appropriate choice of colors X, Y and Z .

We assume that an $m \times n$ rectangle is the smallest non-tileable rectangle satisfying the prescribed parity conditions. By the induction hypothesis, the $m \times (n-2)$ rectangles are thus tileable. It remains to prove that $\mathcal{R}_{m \times 2}^{|\mathfrak{B}|}$ with $m \geq 1$ and $\mathcal{R}_{m \times 2}^{\mathfrak{A}}$ with $m \geq 2$ are also $\mathcal{T}$ -tileable. To do this, we can use two periodic patterns of type $\mathfrak{A}$ and of sizes 2×2 and 3×2 (Fig. 11). These patterns, repeated vertically and cut appropriately, tile the vertical strip $\mathcal{R}_{m \times 2}^{\mathfrak{A}}$ for all $m \geq 2$ ($m \geq 1$ in the case of the vertical strip of $|\mathfrak{B}|$ -type).

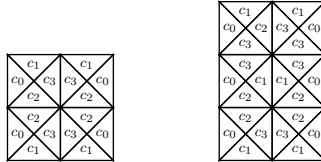


Fig. 11: Examples of patterns of type $\mathfrak{A}$ to tile the vertical strip $\mathcal{R}_{m \times 2}^{|\mathfrak{B}|}$ or $\mathcal{R}_{m \times 2}^{\mathfrak{A}}$ with colors interchanged if required.

Therefore, as the $m \times (n-2)$ rectangles are tileable according to the parities of m and n by assertions from (i) to (iv), and $\mathcal{R}_{m \times 2}^{\mathfrak{A}}$ and $\mathcal{R}_{m \times 2}^{|\mathfrak{B}|}$ are also tileable, concatenating the appropriate rectangles, up to rotation, yields a tiling of the $m \times n$ rectangle. This contradicts our hypothesis that it is the smallest non-tileable rectangle. Thus, all types of rectangle are tileable under the parity conditions on m and n stated in Proposition 1. Finally, it remains to consider the base cases for which such a decomposition is no longer possible ($m \times 1, m \times 2$

or $m \times 3$ depending on the type of rectangle) and to show that each of them admits a $\mathcal{T}$ -tiling under the appropriate parity condition on m .

Proposition 1 immediately yields the following result.

Corollary 1. *Let $\mathcal{T}$ be the tile set. For all $m, n \geq 1$, there exists at least one type of rectangle of size $m \times n$ that admits a $\mathcal{T}$ -tiling.*

In fact, the various decompositions presented above make this result immediate. Indeed, for all $m, n \geq 2$, and thanks to Lemma 1, we get the following rectangles by removing rows and columns:

$$\mathcal{R}_{m \times n}^{\mathfrak{A}} \longrightarrow \mathcal{R}_{m \times (n-1)}^{\overline{\mathfrak{B}}} \text{ or } \longrightarrow \mathcal{R}_{(m-1) \times n}^{|\mathfrak{B}|} \longrightarrow \mathcal{R}_{(m-1) \times (n-1)}^{\mathfrak{C}}$$

Thus, regardless of the parities of the integers $m, n \geq 1$, we can always construct a $\mathcal{T}$ -tiling of a rectangle of at least one type. More precisely:

- If m and n are both even, then $\mathcal{R}_{m \times n}^{\mathfrak{A}}$ admits a $\mathcal{T}$ -tiling.
- If m is even and n is odd, then $\mathcal{R}_{m \times n}^{\overline{\mathfrak{B}}}$ admits a $\mathcal{T}$ -tiling.
- If m is odd and n is even, then $\mathcal{R}_{m \times n}^{|\mathfrak{B}|}$ admits a $\mathcal{T}$ -tiling.
- If m and n are both odd, then $\mathcal{R}_{m \times n}^{\mathfrak{C}}$ admits a $\mathcal{T}$ -tiling.

4 From tilings to parity: existence of an invariant

We now establish the necessary conditions on m and n by counting the triangles of each color in two different ways. This yields a parity invariant. To do this, we need to introduce new notation.

Let $\mathcal{X}_{\mathcal{T}}$ be the color set of $\mathcal{T}$ and i an element of $\mathcal{X}_{\mathcal{T}}$. We denote by $\mathcal{N}_{\Delta}(i)$ the number of times the color $i \in \mathcal{X}_{\mathcal{T}}$ appears in the tiling (when this tiling exists). $\mathcal{N}_{\Delta_{\partial}}(i)$ corresponds to the number of exterior triangles of color $i \in \mathcal{X}_{\mathcal{T}}$ on the boundary of the rectangle. $\mathcal{N}_{\Delta_{\text{int}}}(i)$ corresponds to the number of interior triangles of color $i \in \mathcal{X}_{\mathcal{T}}$ that are not on the edges of the rectangle. If required, we will use $\mathcal{N}_{\Delta}^{\mathfrak{A}}(i)$ (resp. $\mathcal{N}_{\Delta}^{|\mathfrak{B}|}(i)$, $\mathcal{N}_{\Delta}^{\overline{\mathfrak{B}}}(i)$, $\mathcal{N}_{\Delta}^{\mathfrak{C}}(i)$) to specify which type of rectangle is involved.

Lemma 2. *For each rectangle type, if $\mathcal{R}_{m \times n}$ is $\mathcal{T}$ -tileable, then for all $i \in \mathcal{X}_{\mathcal{T}}$, $\mathcal{N}_{\Delta}(i) = mn$ and $\mathcal{N}_{\Delta}(i) = \mathcal{N}_{\Delta_{\text{int}}}(i) + \mathcal{N}_{\Delta_{\partial}}(i)$, with $\mathcal{N}_{\Delta_{\text{int}}}(i) \in 2\mathbb{N}$ and $\mathcal{N}_{\Delta_{\partial}}(i) \in \{0, m, n, 2m, 2n\}$.*

Proof. For every type of rectangle of size $m \times n$, if the rectangle is $\mathcal{T}$ -tileable, then its tiling contains exactly mn tiles. Hence, there are mn colored triangles of each color i of $\mathcal{X}_{\mathcal{T}}$. We thus have $\mathcal{N}_{\Delta}(i) = mn$. Moreover, we can also count the number of triangles of the same color as the sum of $\mathcal{N}_{\Delta_{\text{int}}}(i)$ and $\mathcal{N}_{\Delta_{\partial}}(i)$ for all $i \in \mathcal{X}_{\mathcal{T}}$. Therefore, we have the following equality:

$$\forall i \in \mathcal{X}_{\mathcal{T}}, \quad \mathcal{N}_{\Delta}(i) = \mathcal{N}_{\Delta_{\text{int}}}(i) + \mathcal{N}_{\Delta_{\partial}}(i) = mn$$

If the tiling is valid, then necessarily each interior triangle shares its base with another interior triangle of the same color. $\mathcal{N}_{\Delta_{\text{int}}}(i)$ is thus even. Furthermore, the number of exterior triangles depends on the chosen type of rectangle and on the color i of $\mathcal{X}_{\mathcal{T}}$. Thus, depending on whether the fixed color i appears on a side of the rectangle, we then obtain the following cases:

- For a valid tiling of $\mathcal{R}_{m \times n}^{\mathfrak{A}}$, $\mathcal{N}_{\Delta_{\partial}}^{\mathfrak{A}}(i) \in \{0, 2m, 2n\}$
- For a valid tiling of $\mathcal{R}_{m \times n}^{|\mathfrak{B}|}$, $\mathcal{N}_{\Delta_{\partial}}^{|\mathfrak{B}|}(i) \in \{0, n, 2m\}$
- For a valid tiling of $\mathcal{R}_{m \times n}^{\overline{\mathfrak{B}}}$, $\mathcal{N}_{\Delta_{\partial}}^{\overline{\mathfrak{B}}}(i) \in \{0, m, 2n\}$
- For a valid tiling of $\mathcal{R}_{m \times n}^{\mathfrak{C}}$, $\mathcal{N}_{\Delta_{\partial}}^{\mathfrak{C}}(i) \in \{m, n\}$

Thus, we have the following relation:

$$\forall i \in \mathcal{X}_{\mathcal{T}}, \quad \mathcal{N}_{\Delta}(i) = \underbrace{\mathcal{N}_{\Delta_{\partial}}(i)}_{\in \{0, m, n, 2m, 2n\}} + \underbrace{\mathcal{N}_{\Delta_{\text{int}}}(i)}_{\in 2\mathbb{N}} = mn$$

□

Corollary 2. *If there exists a $\mathcal{T}$ -tiling, then for all $i \in \mathcal{X}_{\mathcal{T}}$, $\mathcal{N}_{\Delta}(i) \equiv \mathcal{N}_{\Delta_{\partial}}(i) \pmod{2}$.*

Proof. By Lemma 2, we can deduce that the sum of $\mathcal{N}_{\Delta_{\partial}}(i)$ and $\mathcal{N}_{\Delta_{\text{int}}}(i)$ has the same parity as $\mathcal{N}_{\Delta_{\partial}}(i)$ because $\mathcal{N}_{\Delta_{\text{int}}}(i)$ is even. Therefore, for all $i \in \mathcal{X}_{\mathcal{T}}$, $\mathcal{N}_{\Delta}(i) \equiv \mathcal{N}_{\Delta_{\partial}}(i) \pmod{2}$. □

From these considerations, we are going to prove the following proposition (necessary conditions of Theorem 1).

Proposition 2. *Let $\mathcal{T}$ be the tile set under consideration.*

- (i) *Let $m, n \geq 2$. If $\mathcal{R}_{m \times n}^{\mathfrak{A}}$ is $\mathcal{T}$ -tileable, then $mn \equiv 0 \pmod{2}$.*
- (ii) *Let $m \geq 2$ and $n \geq 1$. If $\mathcal{R}_{m \times n}^{\overline{\mathfrak{B}}}$ is $\mathcal{T}$ -tileable, then $m \in 2\mathbb{N}$.*
- (iii) *Let $m \geq 1$ and $n \geq 2$. If $\mathcal{R}_{m \times n}^{|\mathfrak{B}|}$ is $\mathcal{T}$ -tileable, then $n \in 2\mathbb{N}$.*
- (iv) *Let $m, n \geq 1$. If $\mathcal{R}_{m \times n}^{\mathfrak{C}}$ is $\mathcal{T}$ -tileable, then $m \equiv n \pmod{2}$.*

Proof. We have proved, by Lemma 2, that for every $i \in \mathcal{X}_{\mathcal{T}}$ and for each rectangle type, if $\mathcal{R}_{m \times n}$ is $\mathcal{T}$ -tileable then, for all $i \in \mathcal{X}_{\mathcal{T}}$, $\mathcal{N}_{\Delta}(i) = \mathcal{N}_{\Delta_{\text{int}}}(i) + \mathcal{N}_{\Delta_{\partial}}(i) = mn$, and Corollary 2 gives $\mathcal{N}_{\Delta}(i) \equiv \mathcal{N}_{\Delta_{\partial}}(i) \pmod{2}$. For the $\mathfrak{A}$ -type and $\mathfrak{B}$ -type, a color absent from the boundary of the rectangle gives $mn \equiv 0 \pmod{2}$. Hence, m and n cannot be both odd. Moreover, when a color occurs on exactly one vertical edge of the rectangle of the $\overline{\mathfrak{B}}$ -type, we get $mn \equiv m \pmod{2}$. A color absent from its boundary also gives $mn \equiv 0 \pmod{2}$. Combining these two congruences, we get $m \equiv 0 \pmod{2}$. Consequently, m must be even. Similarly, for the $|\mathfrak{B}|$ -type, a color that occurs on exactly one horizontal edge of the rectangle gives $mn \equiv n \pmod{2}$ and a color absent from its boundary gives also $mn \equiv 0 \pmod{2}$. Hence, n must also be even. Finally, for the $\mathfrak{C}$ -type, each edge of the rectangle is represented by just one color. We choose a color on a vertical edge and another on a horizontal edge. We thus have

$mn \equiv n \pmod{2}$ and $mn \equiv m \pmod{2}$. Thus, m and n must have the same parity. This proves all four assertions of the previous Proposition 2. $\square$

By Propositions 1 and 2, we deduce Theorem 1 stated in Section 2. We can then deduce the following result on the squares of edge length $n \geq 1$.

Corollary 3. *Let $\mathcal{T}$ be the tile set under consideration and let $\mathcal{R}_n$ be a square of edge length $n \geq 1$.*

- (i) *Let $n \geq 2$. $\mathcal{R}_n^{\mathfrak{A}}$, $\mathcal{R}_n^{\overline{\mathfrak{B}}}$ and $\mathcal{R}_n^{|\mathfrak{B}|}$ are $\mathcal{T}$ -tileable $\Leftrightarrow n \equiv 0 \pmod{2}$.*
- (ii) *For all $n \geq 1$, $\mathcal{R}_n^{\mathfrak{C}}$ is $\mathcal{T}$ -tileable.*

We have thus established necessary and sufficient conditions on m and n for tiling each type of rectangle. Our counting argument relied on the equality $\mathcal{N}_{\Delta}(i) = mn$, with four colors, every tile contains each color exactly once. When $|\mathcal{X}_{\mathcal{T}}| > 4$, a tile need not contain the chosen color i , so this equality is no longer true in general. A natural question is now whether rectangles of every type and all dimensions $m, n \geq 2$ can be tiled using a larger set of colors. We therefore determine the minimum number of colors required to guarantee the existence of such tilings.

5 Minimum number of colors of $\mathcal{X}_{\mathcal{T}}$ required to tile rectangles of all types

The previous results have shown that if $|\mathcal{X}_{\mathcal{T}}| = 4$, then four colors do not suffice to tile rectangles of every type for all $m, n \geq 2$. This finding gives us a lower bound on the minimum number of colors of $\mathcal{X}_{\mathcal{T}}$ required. At least 5 colors are required in the tile set. Hence,

$$\min |\mathcal{X}_{\mathcal{T}}| \geq 5.$$

Recall that the tile set $\mathcal{T}$ contains all tiles whose four edges are assigned pairwise distinct colors from $\mathcal{X}_{\mathcal{T}}$.

In this section, we prove that $\min |\mathcal{X}_{\mathcal{T}}| \leq 5$. This means that five colors are sufficient to tile each type of $m \times n$ rectangles and for all $m, n \geq 2$. From this result, we will be able to deduce Theorem 2.

To do this, we therefore need to deal with the impossible cases when $|\mathcal{X}_{\mathcal{T}}| = 4$:

- m and n both odd for the $\mathfrak{A}$ -type;
- n odd and regardless of the parity of m for the $|\mathfrak{B}|$ -type;
- m odd and regardless of the parity of n for the $\overline{\mathfrak{B}}$ -type;
- m odd and n even or vice versa for the $\mathfrak{C}$ -type.

Remark 3. The case of $|\mathfrak{B}|$ -type can be deduced from the $\overline{\mathfrak{B}}$ -type by quarter-turn rotation and interchanging the role of m and n . Therefore, we will only consider the case of $\overline{\mathfrak{B}}$.

The main idea is to show that, for each type of $m \times n$ rectangle, we can add a row or column whose colors match with the edges of rectangles already tiled. To do this, we use a fifth color. Let $\mathcal{X}_{\mathcal{T}}$ be a color set such that $|\mathcal{X}_{\mathcal{T}}| = 5$. Denote $\mathcal{X}_{\mathcal{T}} = \{c_0, c_1, \dots, c_4\}$. First, we prove a simple result for vertical strips that will be used later.

Proposition 3. *Let $\mathcal{T}$ be a tile set such that $|\mathcal{X}_{\mathcal{T}}| = 5$. Then for all $m \geq 2$, there exists a $\mathcal{T}$ -tiling for the vertical strips $\mathcal{R}_{m \times 1}^{\overline{\mathfrak{B}}}$ and $\mathcal{R}_{m \times 1}^{\mathfrak{C}}$.*

Proof. By Theorem 1, we know there exists a tiling for the vertical strip of type $\overline{\mathfrak{B}}$ when m is even. It remains to show that this result is also valid when $m \geq 2$ is odd. To do this, we can build a pattern $\mathcal{R}_{2 \times 1}^{\overline{\mathfrak{B}}}$. This pattern has the same color c_1 on its north and south edges. Hence, when $m \geq 2$ is odd, we can repeat this pattern $\lfloor m/2 \rfloor$ times to tile the first $m - 1$ rows of the vertical strip composed of $2\lfloor m/2 \rfloor$ tiles (Fig. 12). We then replace the north-edge color c_1 of the $(m - 1)$ th tile with the fifth color which has not yet been used (for instance c_3 as shown in Fig. 12). We then add one final tile whose south-edge color is the new north-edge color of the vertical strip and whose north-edge color is now c_1 (Fig. 12). Therefore, with five colors, there exists a tiling of the vertical strip when $m \geq 2$ is also odd. Thus, for all $m \geq 2$, $\mathcal{R}_{m \times 1}^{\overline{\mathfrak{B}}}$ is $\mathcal{T}$ -tileable (Fig. 12).

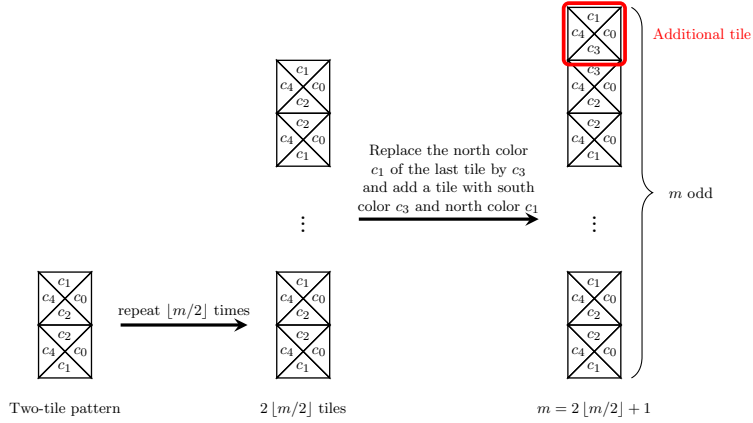


Fig. 12: Construction of a vertical strip of odd height $m \geq 2$. The two-tile pattern is repeated $\lfloor m/2 \rfloor$ times. The upper color c_1 is then replaced by c_3 , allowing one final tile to be added.

Moreover, we also know by Theorem 1 that there exists a $\mathcal{T}$ -tiling for the $\mathfrak{C}$ -type rectangle, when $m \geq 1$ and $n \geq 1$ have the same parity. The vertical strip is thus tileable, in particular, when $m \geq 1$ is odd. We are going to prove that it is also tileable when m is even, adding a fifth color. The proof is easy. When $m \geq 1$ is even, it suffices to repeat $m/2$ times the same two-tile pattern (Fig. 13). It then remains to replace the north color c_1 of the last tile by the fifth color that does not appear in the tiling. In our case, the fifth color is c_3 . Therefore, we have proved that there exists a $\mathcal{T}$ -tiling for the vertical strip $\mathcal{R}_{m \times 1}^{\mathfrak{C}}$ when $m \geq 2$ is even (Fig. 13).

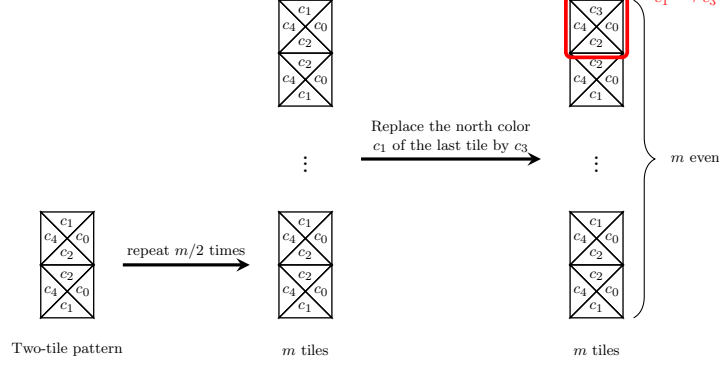


Fig. 13: Construction of a vertical strip of even height $m \geq 2$. The two-tile pattern is repeated $m/2$ times. The upper color c_1 is then replaced by c_3 .

□

Proposition 3 will be used to prove the following proposition which deals with all cases that are impossible with four colors.

Proposition 4. Let $\mathcal{T}$ be the tile set such that $|\mathcal{X}_{\mathcal{T}}| = 5$.

- (i) If $m, n \geq 2$ are both odd, then $\mathcal{R}_{m \times n}^{\mathfrak{A}}$ is $\mathcal{T}$ -tileable.
- (ii) If $m \geq 2$ is odd and $n \geq 1$, then $\mathcal{R}_{m \times n}^{\overline{\mathfrak{B}}}$ is $\mathcal{T}$ -tileable.
- (iii) If $n \geq 2$ is odd and $m \geq 1$, then $\mathcal{R}_{m \times n}^{|\mathfrak{B}|}$ is $\mathcal{T}$ -tileable.
- (iv) If $m, n \geq 1$ are different parities, then $\mathcal{R}_{m \times n}^{\mathfrak{C}}$ is $\mathcal{T}$ -tileable.

Proof. Let $\mathcal{T}$ be the tile set such that $|\mathcal{X}_{\mathcal{T}}| = 5$. We distinguish several cases.

Cases 2 and 3: $\overline{\mathfrak{B}}$ and $|\mathfrak{B}|$ -type. We start with assertion (ii) and then we will deduce assertion (iii). We know, by Theorem 1, that $\mathcal{R}_{m \times n}^{\mathfrak{A}}$ is $\mathcal{T}$ -tileable, in particular when m is odd and n is even, using only four colors in $\mathcal{X}_{\mathcal{T}}$. It then suffices to add a vertical strip $\mathcal{R}_{m \times 1}^{\overline{\mathfrak{B}}}$ with the same color on its west edge as on the east edge of $\mathcal{R}_{m \times n}^{\mathfrak{A}}$, and the fifth color on its east edge. This vertical strip is also $\mathcal{T}$ -tileable by Proposition 3. By concatenation, we get a $\mathcal{T}$ -tiling of $\mathcal{R}_{m \times n'}^{\overline{\mathfrak{B}}}$ with m odd and $n' = n + 1$ also odd. Hence, when m and n are both odd, then $\mathcal{R}_{m \times n}^{\overline{\mathfrak{B}}}$ is $\mathcal{T}$ -tileable.

Furthermore, by applying a quarter-turn rotation and switching the roles of m and n , we get a tiling for the $\mathcal{R}_{m \times n}^{|\mathfrak{B}|}$ when m and n are also both odd. It just remains to verify, for the $\overline{\mathfrak{B}}$ -type that there exists a $\mathcal{T}$ -tiling when m is odd and n is even, since by rotation one will be able to deduce the similar result for the $|\mathfrak{B}|$ -type. In this case, we know there exists a $\mathcal{T}$ -tiling for $\mathcal{R}_{m \times n}^{\mathfrak{A}}$ when m is odd and n even. It then suffices to replace the east-edge color of each tile in the last column by the fifth color, which has not yet been used. Thus, we get a $\mathcal{T}$ -tiling for the $\mathcal{R}_{m \times n}^{\overline{\mathfrak{B}}}$ with m odd and n even. Therefore, by rotating a quarter turn and

switching the roles of m and n , we have also a $\mathcal{T}$ -tiling for the $\mathcal{R}_{m \times n}^{|\mathfrak{B}|}$ when m is even and n is odd. Hence, the assertions (ii) and (iii) are proved.

Case 1: $\mathfrak{A}$ -type. Since we have used the fifth color only on the east edge of the tiling of $\mathcal{R}_{m \times n}^{\mathfrak{B}}$ when m and n are both odd, it then suffices to replace the west color of the first column, whose tiles use only the original four colors, by this fifth color to obtain a $\mathcal{T}$ -tiling of the $\mathfrak{A}$ -type. This proves the assertion (i).

Case 4: $\mathfrak{C}$ -type. We know, by Theorem 1, there exists a $\mathcal{T}$ -tiling with only four colors when $m \geq 1$ and $n \geq 1$ have the same parity. We can add a vertical strip $\mathcal{R}_{m \times 1}^{\mathfrak{C}}$ which has a west-edge color matching the east-edge color of the $\mathcal{R}_{m \times n}^{\mathfrak{C}}$, and the fifth color on its east edge. Moreover, thanks to Proposition 3, the vertical strip $\mathcal{R}_{m \times 1}^{\mathfrak{C}}$ is tileable with five colors for $m \geq 1$, in particular when m is even. Hence, we get a $\mathcal{T}$ -tiling for the $\mathcal{R}_{m \times n'}^{\mathfrak{C}}$, when m is even and $n' = n + 1$ is odd because n has the same parity as m . Moreover, by rotating a quarter turn and switching the roles of m and n , we get another tiling when m is odd and n' is even. In conclusion, for all $m \geq 1$ and $n \geq 1$ of different parities, there exists a $\mathcal{T}$ -tiling. The assertion (iv) is proved. $\square$

Theorem 1 and the previous Proposition 4 yield the following result.

Corollary 4. *Let $\mathcal{T}$ be the tile set such that $|\mathcal{X}_{\mathcal{T}}| = 5$. Then for all types of rectangle and for all $m, n \geq 2$ there exists a $\mathcal{T}$ -tiling.*

Remark 4. This finding is also valid for the $\mathfrak{C}$ -type when $(m, n) = (1, 1)$, for the $|\mathfrak{B}|$ -type when $(m, n) = (1, 2)$ and for the $\mathfrak{B}$ when $(m, n) = (2, 1)$.

Hence, we can deduce from Corollary 4 that five colors suffice to tile each type of rectangle for all $m, n \geq 2$. We thus have:

$$\min |\mathcal{X}_{\mathcal{T}}| \leq 5$$

In conclusion, the minimum number of colors of $\mathcal{X}_{\mathcal{T}}$ required to tile rectangles of every type for all dimensions $m, n \geq 2$ is

$$\min |\mathcal{X}_{\mathcal{T}}| = 5.$$

Theorem 2 has now been proven. As noted in Remark 4, the tilability result also extends to some rectangles with $m = 1$ or $n = 1$, depending on their type.

6 Towards other friendly problems

In this paper, we have seen that when $|\mathcal{X}_{\mathcal{T}}| = 4$ or 5, we do not use all the tiles of $\mathcal{T}$ to tile the different types of $m \times n$ rectangles. A natural question is:

- *What is the minimum number of tiles of $\mathcal{T}$ required to tile all types and sizes of rectangles?*

A possible approach to determine $\min |\mathcal{T}|$ consists of using the smallest periodic patterns to construct new tilings as in the proof of the sufficient conditions of Theorem 1. Hence, once these smallest periodic patterns have been identified, we will be able to determine this minimum number of tiles of $\mathcal{T}$.

Let us now consider a tile set $\mathcal{T}$ such that $|\mathcal{X}_{\mathcal{T}}| \leq 4$. Therefore, a tile of $\mathcal{T}$ does not necessarily have to contain four different colors. The question is:

- *Is it possible to determine whether a $\mathcal{T}$ -tiling exists for the $m \times n$ rectangles with monochromatic edges?*

A first approach would be to study the problem for the different types of tiles using only one, two, three or four colors on their tiles. For instance, with tiles that have exactly four colors, we have seen that a tiling does not always exist. Hence, when $|\mathcal{X}_{\mathcal{T}}| \leq 4$, we need to consider another type of tiles that could play the role of a fifth color...

The initial problem presented in this paper could be extended to the third dimension. Indeed, we can consider a Wang cube instead of a Wang tile. A Wang cube has been described by Culik and Kari [22]. In our context, a Wang cube is a unit cube which has six different colors on its faces. Let $\mathcal{T}$ be a set of Wang cubes under consideration. All the permutations of the colors are possible. We thus have $|\mathcal{T}| = 6! = 720$ possible Wang cubes. The question is:

- *For all integers $i, j, k > 0$, is it possible to tile an $i \times j \times k$ rectangular cuboid with these Wang cubes so that each of its six boundary faces is monochromatic?*

We believe it would be possible to adopt an approach similar to that used for the 2D problem presented in this article. We could make tilings using a $2 \times 2 \times 2$ periodic cube or $2 \times 2 \times 3$ periodic rectangular cuboid up to rotation, and if required, appropriate terminal surfaces parallel to the coordinate planes (xOy) , (xOz) or (yOz) could then be removed while preserving the monochromatic face conditions. Furthermore, if a $\mathcal{T}$ -tiling exists, we also have a relationship similar to that of the 2D problem:

$$\forall c \in \mathcal{X}_{\mathcal{T}}, \quad \mathcal{N}_{\square}(c) = \mathcal{N}_{\square_{\partial}}(c) + \mathcal{N}_{\square_{\text{int}}}(c).$$

$\mathcal{N}_{\square}(c)$ is the number of square faces of color c that is equal to product ijk . $\mathcal{N}_{\square_{\partial}}(c)$ is the number of square faces of color c of the $i \times j \times k$ rectangular cuboid. This is in fact the area of rectangular face of the $i \times j \times k$ rectangular cuboid and which may be zero when the color does not appear on the exterior face of the rectangular cuboid. Finally, $\mathcal{N}_{\square_{\text{int}}}(c)$ is the number of interior square faces of the $i \times j \times k$ rectangular cuboid. This number is always even. Hence:

$$\forall c \in \mathcal{X}_{\mathcal{T}}, \quad \mathcal{N}_{\square}(c) \equiv \mathcal{N}_{\square_{\partial}}(c) \pmod{2}.$$

For example, for the $i \times j \times k$ rectangular cuboid with six different colors on its faces, we get a contradiction when exactly two of these dimensions are odd. Take i and j odd and k even for instance, then the product ijk is even whereas the product ij is odd. This does not satisfy the relationship mentioned above. Hence, if a $\mathcal{T}$ -tiling exists for $i \times j \times k$ rectangular cuboid with six different colors on these faces, then the integers i, j and k are all odd or at least two of them are even. We conjecture that this necessary condition is also sufficient. It seems that the construction method presented in this paper, based on repeating small patterns and cutting along rows or columns, could easily be extended to three dimensions.

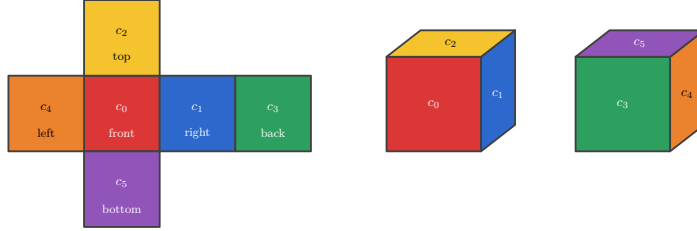


Fig. 14: Example of a Wang cube with six different colors.

Finally, the same counting argument could extend to a d -dimensional rectangular box of dimensions $n_1 \times \dots \times n_d$, with $n_i > 0$ for every i , tiled by d -dimensional unit Wang hypercubes carrying the $2d$ colors exactly once. If its $2d$ boundary facets, of $(d-1)$ -dimension, are monochromatic and have pairwise distinct colors, any tiling should verify the following relation:

$$\forall i \in \{1, \dots, d\}, \quad \prod_{j=1}^d n_j \equiv \prod_{\substack{j=1 \\ j \neq i}}^d n_j \pmod{2}.$$

Denote $N = \prod_{j=1}^d n_j$, the previous congruences can then be written as:

$$\forall i \in \{1, \dots, d\}, \quad N \equiv \frac{N}{n_i} \pmod{2} \iff \frac{N}{n_i}(n_i - 1) \equiv 0 \pmod{2}.$$

If all n_i are odd, then these previous congruences hold. Otherwise, for any $i \in \{1, \dots, d\}$ such that n_i is even, at least one other even dimension is required. Hence, the necessary condition established in three dimensions extends to dimension d : if a tiling of a d -dimensional rectangular box by unit Wang hypercubes exists, then either all its dimensions are odd or at least two are even. In particular, when $d = 2$, we have the necessary condition for rectangle of $\mathfrak{C}$ -type, and when $d = 3$, we get again the condition mentioned above. Therefore, a natural direction for further investigation will be to determine whether this necessary condition is also sufficient for the existence of tilings in dimension $d \geq 3$.

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