Study of the potential of problems to practice a research activity in mathematics at elementary school in French-speaking Switzerland

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Problem solving is one of the essential elements of the curriculum for students in French-speaking Switzerland. Currently, it is at the core of the priority objectives in mathematics teaching and a chapter "help for problem solving" is dedicated to it for each teaching cycle. The aim of this paper is to study the potential of problems to practice a research activity in mathematics for elementary students. After having defined from an epistemological point of view the notions of mathematical activity and problem, 13 mathematical statements proposed in this specific chapter for 10–11-year-old students were analyzed. The results show that these statements are not problems in the sense of the definition given in this paper and the mathematical activity potentially produced would however be limited from the point of view of the knowledge mobilized through the processes of experimentation, formulation, and validation.

Keywords: Mathematical activity, mathematical problem, epistemology, syntax, semantics.

Introduction

In view of its federal structure, Switzerland has delegated to the provinces (called "cantons") all matters related to education. In 2006, a new curriculum (called "Plan d'Etudes Romand", hereafter PER) was progressively introduced in order to harmonize the school systems of the French and Italian speaking parts of the country (about eight cantons among the twenty-six). A multi-provincial conference (CIIP) was then given the mandate to write teaching materials (hereafter MER) in accordance with the PER. Starting in 2016, the different provinces were able to progressively introduce these new resources according to their own timetable at each level of schooling. It should be noted that the first provinces are introducing the elements of 7H (Grade 5, students aged 10-11 years) in this school year 2022-2023. This is a reference resource that teachers, with respect to their pedagogical freedom, can nevertheless complete according to their needs. However, these MERs are compulsory at the institutional level, particularly because they have been developed, constructed and written in order to work on the PER's disciplinary content. As regards Cycle II (students aged 8 to 12), the structure of the MERs follows the structure of the PER (the thematic axes relating to numbers, operations, geometry, measures, etc.) while giving importance to cross-curricular elements such as problem solving. This quest for compliance has led the drafters of the MERs to include an entire section devoted to problem solving assistance ("*Aide à la résolution de problème*" in French, hereafter ARP). Like the other thematic axes, this ARP section is, for the 7H (10–11-year-old students), subdivided into four chapters (appropriating a mathematical problem, using research strategies, verifying the answer to a problem, communicating the result of one's research) with the particularity, however, of proposing both specific statements and links to statements from the other thematic axes. The stated issue of this ARP section is to set up a problem-solving teaching:

> Problem solving, which is at the heart of the priority aims of mathematics teaching from cycle 1 to cycle 3 (students aged 4 to 15), is also a source of difficulty for many students. Clearly, it is not enough to let students look for problems and then correct them using the students who have succeeded in doing so, so that everyone learns to solve problems. It is necessary to set up a real teaching of problem solving. That's what the ARP part is all about. (author's translation, CIIP, 2023)

It is this issue that we propose to question here. To do so, we define, in a first part, the different objects used in our research, related to the mathematical activity and to the notion of problem, through epistemological views. From these definitions, in a second part, we will explain our research questions and method. Finally, in the third part, we present the results of our analysis of the problems of the 7H ARP section (10–11-year-old students).

Theoretical framework

Practice of mathematical activity

Mathematical activity is, for mathematicians and mathematics educators, closely linked to problem solving (e.g., Brousseau, 1997; Chevallard, 1998; Halmos, 1980; Thurston, 1994). Doing mathematics is therefore to solve problems by accepting scientific responsibility for the question of truth and the need for proof in relation to a given problem (Da Ronch, 2022; Gandit, 2008). This activity should thus allow the implementation of different solving processes related to experimentation, formulation, and validation (Brousseau, 1997). This requires the mobilization of specific knowledge linked to the practice of the mathematical activity. For example, entering into the resolution of a problem by studying particular cases is a knowledge linked to the experimentation process. The formulation of conjectures resulting from the study of these cases, the change of the register of representation of the mathematical objects at stake, sometimes leading to modelling premises, are examples of knowledge linked to the formulation process. Finally, validation is based on hypotheses (e.g., law of excluded middle, law of noncontradiction) and rules of logic (e.g., *modus ponens*, *modus tollens*) which allow, thanks to different mathematical reasoning (e.g., direct implication, contradiction, induction, counterexample...) to enter into the argumentation and proof process and thus to rule on the truth or falsity of a mathematical statement. This knowledge is therefore knowledge related, this time, to the validation process (Da Ronch, 2022). Moreover, validation is only possible if definitions about the manipulated mathematical objects are clearly stated. Indeed, the act of definition is a fundamental process in the formation of concepts and is generally the immediate consequence of the search for the proof or a component of the proof (Ouvrier-Buffet, 2006; Da Ronch, 2022). Thus, the implementation of this knowledge is the articulation between inductive and deductive reasoning inherent to the practice of mathematical activity, and also shows the experimental character of this activity (see, e.g., Gardes & Durand-Guerrier, 2016; Giroud, 2011; Da Ronch, 2022).

Concept of the problem

In order to enter into the practice of mathematical activity, it is necessary to deal with problems that allow access to all the components of this practice. Thus, solving a "real" problem should allow working on experimentation, formulation and validation which are the essential components of this practice. But then, what definition should be given to this concept? In the mathematical community, a problem can be defined according to its syntactic and semantic aspects (Da Ronch, 2022). Indeed, Garey and Johnson (1979) structurally characterize the formulation of a mathematical problem as a couple formed by a general question and a set of instances[1].

> For our purposes, a problem will be a general question to be answered, usually possessing several parameters, or free variables, whose values are left unspecified. A problem is described by giving: a general description of all its parameters, and a statement of what properties the answer, or solution, is required to satisfy. An instance of a problem is obtained by specifying particular values for all the problem parameters. (Garey & Johnson, 1979*,* p. 4)

Furthermore, in order to enter into the practice of mathematical activity, it is necessary that the problem is semantically interesting to solve. That is to say, the problem must contain a sufficiently significant epistemological quantity to echo other related problems. This means that the problem itself is not an isolated problem and is therefore not semantically poor. In general, it is from changes in the formulation of the problem considered (scope of the question and instances) that other related mathematical problems appear. The proximity with other problems can also be found in the mobilization of the same proof invariants that allow them to solve (Da Ronch, 2022; Giroud, 2011). For example, let us consider the following statement (see Figure 1): *Given a rectangular grid of size p×q and a collection of 1×2 dominoes, is there a tiling of the grid of size p×q with these dominoes when a unit square of the grid is removed?*

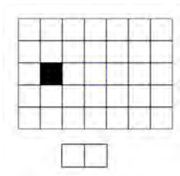


Figure 1: Example of a 5x7 grid with a "hole" and dominoes

Here, we notice first there is a general scope question and specified instances (integers *p* and *q*, dominoes, a unit square) that allow to validate the syntactic aspect. Then, the semantic aspect is also validated since it holds a significant epistemological quantity which allows to say that it is semantically interesting to solve. Indeed, it is close to other related problems thanks to the modification of its instances (*L*-triminos, several removed unit squares, shape of the surface to be paved...) and remains, in a general way (polymino), still open in the current mathematical research (see for example Grenier & Payan, 1998; 2002). Moreover, these authors show that there are also proofs of existence of tilings using other registers of representation of the object at stake like bipartite graphs with the notion of Hamiltonian path. This echoes other semantically interesting neighbouring problems such as the *problems of existence of a path or a Hamiltonian cycle in a graph* or that of *the travelling salesman* (Karp, 1972). Finally, Grenier and Payan (1998) show the interest of this type of

[1] A problem can be described as an injunctive sentence formulating a demand or as a mathematical statement of a conjecture. In any case, they can always be reformulated as a question (Da Ronch, 2022).

problem in the discrete domain of mathematics to work on argumentation and proof in class through the different components of mathematical activity (experimentation, formulation, validation).

Thus, in order to determine whether a proposed "statement" is a problem, it is necessary to check that it is syntactically correct and semantically rich. To do this, the concept of variable of research (Grenier & Godot, 2004) will be useful to rule on this first aspect. Indeed, a variable of research is a parameter of the problem that is not initially set by the teacher, but it is up to the student to set the values of this variable. For instance, if one wants to determine whether a rectangle of size $p \times q$ can be tiled by 1×2 dominoes, the variable of research here is the size of the grid, which is not fixed upstream by the teacher, and it is therefore up to the pupil to appropriate it, that is to say to fix its values himself in the process of solving the problem. Therefore, it is the identification of these variables of research that will be used as indicators within our research method to validate the syntactic aspect. If the syntactic aspect is validated, the mathematical analysis will allow to validate (or not) the semantic aspect and thus to determine if a proposed "statement" is a problem according to our definition.

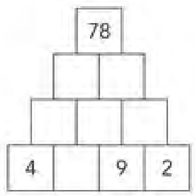
Research questions and method

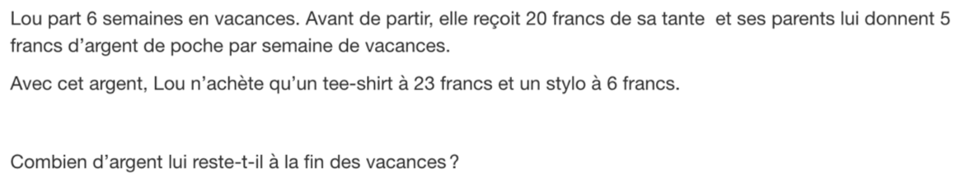
In terms of these definitions concerning mathematical activity and the concept of problem, the two research questions are as follows: (1) Are the statements proposed in ARP mathematical problems? (2) What components of mathematical activity can be mobilized through these statements?

A corpus including the 13 statements appearing exclusively in the ARP chapter (the statements shared between ARP and other thematic axes have not been analyzed) has been constituted. These statements have for the moment the status of "candidate-problems" and our method consists in applying the epistemological criteria (syntactic and semantic) to each of these "candidate-problems". The existence of a variable of research is a criterion that allows us to identify, from a syntactic point of view, whether the question posed is general or not. The second criterion, relating to the epistemological quantity of the problem, determines whether the proposed candidate-problem is semantically rich. The epistemological quantity of a problem is characterized by the formulation of its question and its scope, as well as by its instances and the invariants of proof mobilized to solve it. Satisfying the first criterion becomes a necessary but not sufficient condition for the candidate-problem to be considered as a problem in the sense we have defined it. The criteria are coded in a binary way: 1 if the criterion is satisfied, 0 otherwise. Each candidate-problem are coded by a couple (a,b), with a and b numbers equal 0 or 1; a is for the syntactic criteria (existence of variable of research) and b for semantic criteria (links with other problems, epistemological quantity). There were also four possibilities for each statement: (0,0), (0,1), (1,0), (1,1). For this phase of our research, the coding was carried out jointly and not independently, so that the coders could agree on the meaning of the criteria and coordinate the values assigned. The method relating to research question 2 consists in producing an a priori analysis of the situation by describing and analyzing each phase of the potential mathematical activity of the pupils, i.e., the experimentation, formulation, and validation phases (Brousseau, 1997).

Results

The results of our epistemological analysis of the statements of the 13 "candidate-problems" show that only four problems were coded (1,0) and the rest were codded (0,0). As a result, among the 13 candidate-problems, 9 are not, from a syntactic point of view, a mathematical problem. Indeed, the non-existence of a variable of research shows that the question asked does not have a general scope. They are for example "Pyramide de nombres" and "Argent pour les vacances" (Figure 2a and 2b).




Lou part 6 semaines en vacances. Avant de partir, elle reçoit 20 francs de sa tante et ses parents lui donnent 5 francs d'argent de poche par semaine de vacances.

Avec cet argent, Lou n'achète qu'un tee-shirt à 23 francs et un stylo à 6 francs.

Combien d'argent lui reste-t-il à la fin des vacances ?

Figure 2: Statement of "Pyramides de nombres" [2] (2a) and "Argent pour les vacances" [3] (2b)

Only the statements of the 4 "candidate-problems" thus satisfy the syntactic criteria. All of them belong to the section "Trying out, finding a rule, testing it and, if possible, validating it" of the chapter "Using search strategies" (chapter 2, see introduction). They consist in formulating a general rule to determine the number of iterations or elements constituting a pattern. In this statement (Figure 3), the search variable is "the size of the cross". Some values are mentioned (20, 50, 100) but other possible values are suggested by the presence of suspension points and by the universal character of the question (*any size*, see Figure 3). This one is left to the subject who solves it.

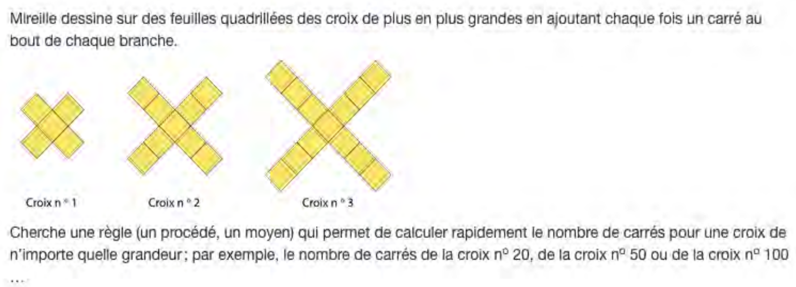


Figure 3: Statement of the candidate-problem "De croix en croix" [4]

Among these four candidate-problems, none of them satisfies the semantic criteria. Indeed, they are isolated, considering the weakness of their epistemological quantity. These statements have similar instances: the pattern is fixed and simple, the regularity is elementary. The scope of their questions is

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[2] Complete the pyramid so that the number of one brick is the sum of the numbers of the two bricks on which it rests.

[3] Lou is going on vacation for 6 weeks. Before leaving, she receives 20 francs from her aunt and her parents give her 5 francs pocket money per week of vacation. With this money, Lou buys a T-shirt for 23 francs and a pen for 6 francs. How much money does she have left at the end of the vacation?

[4] Mireille draws larger and larger crosses on squared paper, adding a square at the end of each branch. Find a rule (a process, a way) that makes it possible to quickly calculate the number of squares for a cross of any size; for example, the number of squares for cross #20, cross #50 or cross #100.

restricted to the study of the only pattern considered. Finally, the resolution mobilizes the same invariant of proof which is based on the principle of recurrence only. Therefore, these statements do not allow the emergence of new related research questions which could enrich the considered candidate-problems from a semantic point of view.

Thus, to answer the first research question, none of the 13 candidate-problems of the ARP satisfies the two criteria, syntactic and semantic, and can therefore be qualified as a mathematical problem, according to the definition given in this text. This lack of a mathematical problem has a significant impact on students' mathematical activity, since problem solving is a necessary condition for the practice of mathematical activity. Indeed, all the components of this practice linked to the process of experimentation, formulation and validation cannot potentially be mobilized in an exhaustive manner. However, these statements exist in the curriculum of the French-speaking Swiss elementary school. It therefore seems relevant to question the potential mathematical activity during their resolution (second research question). In view of the results of the first research question, only the four statements on patterns, satisfying the first epistemological (syntactic) criteria, will be considered.

When solving the problem, students can enter into the process of experimentation, notably by implementing inductive reasoning based on the study of particular cases (e.g., crosses #5, #20, #50, #100, see Figure 3). However, this process is rather guided by the formulation of the statement and leaves very limited scientific responsibility to the students to experiment.

Following these experiments, formulation phases are possible, for example formulating conjectures (e.g., to go from one cross to another, one adds 4 squares; to find the number of squares of any cross, one multiplies by 4 and adds 1). However, the formulation of the statement takes charge of defining the mathematical objects necessary for the students' mathematical activity: the pattern and the construction of the iteration of the pattern. This facilitates the development of a conjecture about the number of squares based on the previous construction step.

The validation of these conjectures can be done by the student, at the local level (e.g., on a finite subset of N) by checking their formulas on particular cases, at the global level by counter-example to refute a formula and by using a generic example to, on the contrary, validate it (Balacheff, 1987), since it is the only type of proof available at this level. Indeed, only arguments based on the principle of recurrence can be used to validate, but the formalization of the proof is not accessible. Moreover, the statement does not ask about the existence and uniqueness of answers.

To answer the second research question, it is thus possible to work on all the components of the mathematical activity but only in a very limited way. Indeed, from a didactic point of view, these statements dealing with patterns are interesting to work on algebraic thinking, in particular the entry in generalization and the introduction of the letter as a variable. On the other hand, the access to an exhaustive practice of mathematical activity is limited with this type of statements (see e.g., Figure 3). For example, in this context, the experimentation is very guided (schematization given of the different steps, particular cases fixed in the statement), the formulation of conjecture is moreover guided by the statement and the validation is limited to the explanation of the formula in relation to the given schemas. Moreover, the act of defining the mathematical objects at stake (patterns) is obviously not left to the students. Thus, this shows that the students' mathematical activity will certainly be incomplete at any level of this practice (experimentation, formulation, validation).

Conclusion and perspectives

Problem solving is one of the essential elements of the curriculum for students in French-speaking Switzerland (cycle 1 to cycle 3, students aged 4 to 15). Currently, it is at the core of the priority objectives in mathematics teaching and a chapter " help for problem solving " is dedicated to it for each teaching cycle.

After having defined from an epistemological point of view the notions of mathematical activity and problem, this paper demonstrates, at the level of cycle 2 and in particular in 7H (students aged 10 to 11), that the mathematical statements proposed in this specific chapter are not problems in the sense of the definition given in this paper. Indeed, in this text a mathematical problem is characterized according to two aspects: syntactic (structure) and semantic (meaning). However, none of the 13 statements studied in this corpus respects both the syntactic criteria in the formulation of a mathematical problem (general question and set of instances) and the semantic criteria which shows that they are epistemologically consistent and therefore not isolated. From a didactic point of view, solving statements that cannot be qualified as problems reduces the student's mathematical activity to an incomplete mathematical activity. On this occasion, the statements in the problem-solving chapter with the greatest potential to engage students in the practice of mathematical activity are those related to patterns. However, from a semantic point of view, they are rather isolated and are not epistemologically consistent. We have shown that the mathematical activity potentially produced would however be limited from the point of view of the knowledge mobilized through the processes of experimentation, formulation, and validation. Thus, the statements proposed in this chapter are not satisfactory to develop a real practice of mathematical activity in class... However, there are many mathematical problems accessible at different levels of education that allow for advanced practice of mathematical activity in the classroom. The Research Situations for the Classroom (Grenier & Godot, 2004) developed from professional discrete mathematics problems are examples (see e.g., Da Ronch et al., 2020, 2021; Da Ronch, 2022; Gravier & Ouvrier-Buffet, 2022; Grenier & Payan, 1998). In the continuity of this research, we will test one of these situations in 7H classes (students aged 10 to 11) in order to demonstrate that a real practice of mathematical activity in class is possible at this level of teacher.

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